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THE PSYCHOLOGY OF PROBLEM SOLVING*

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In a previous article we called attention to some of the differences of opinion concerning the application of algebraic technique to the solution of problems. In the present article we shall take up systematically the questions there suggested, together with others, and present the results of certain investigations which we have made to aid teachers in deciding what to do with problem solving, when to do it, and how to do it.

At the outset we need to distinguish certain types of work all of which might with some justification be called applications of algebraic technique to the solution of problems, but which differ notably in the psychological demands they make of a pupil, in the psychological effects they have upon him, and in their uses in the algebra course. All are different from mere computation, evaluation, reading or making graphs, or the solution of equations already framed.

The more important types are shown below.

Type I. Problems to answer which no explicit equation or formula is needed or supposed to be used.

1. Concerned with knowledge of meanings, *e. g.*, of literal numbers, negative numbers, exponents.

If pencils cost c cents each what will n pencils cost?
Express 248 ft. below sea level if sea level is called 0.
If $a \times a \times a \times a$ is a^4, how will you express the product of a row of n a's?

2. Concerned with knowledge of operations.

What was your average score in a game in which you made these separate scores: -8, $+4$, -6, -2, $+7$?

3. Concerned with combinations of meanings and operations or with other aspects of algebra.

Under what conditions will $a = \frac{1}{a}$?

In $a = p + \frac{q}{r}$ what will be the effect upon a of an increase in r?

*The investigations on which this article is based were made possible by a grant from the Commonwealth Fund.

The value of v_2 for $v_1 = 4$ was omitted from this table by the printer. What do you think it was, approximately?

If	v_1	is	0	v_2	is	4
"	"	"	1	"	"	5.02
"	"	"	2	"	"	7.96
"	"	"	3	"	"	12.98
"	"	"	4	"	"	.....
"	"	"	5	"	"	29.03
"	"	"	6	"	"	40.05

Type II. Problems to answer which an equation or formula is supposed to be used.

A. The equation or formula is one of a group of known formulae.

1. Which of these formulae fits the problem is also known.

   $F = 32° + \frac{9}{5} C$.
   What does 86° F equal on the Centigrade scale?

2. Which of them fits the problem is not known. Pupil must select the formula as well as fit the facts properly to it.

   The meanings of these are known, $v = at$, $v = gt$, $S = \frac{1}{2}at2$, $S = \frac{1}{2}gt2$, $v = \sqrt{2as}$, $v = \sqrt{2gs}$, $g = 32$.
   Neglecting the resistance of the air, how far will a bullet dropped from a height of 5000 ft. fall during the sixth second?

B. The equation or formula is not known, but must be constructed by the pupil.

a. The equation or formula is primarily a rule for all cases of a certain relation.

   Frame equations for finding the dimensions of a rectangle twice as long as it is wide to be of *a* sq. in. area. Find the dimensions when *a* is 10. When *a* is 20. When *a* is 100.

b. The equation (not usually called a formula) is primarily an organization of data to secure the result in one special case.

   A girl wishes to have a rectangular card twice as long as it is wide and 10 sq. in. in area. How long shall she make it?

The psychology of the I-1 type is clear. Such problems, carefully chosen and graded, may be used very helpfully to teach meanings and to test, strengthen, extend and refine knowledge of meanings. Teachers of algebra should study the problem material of this sort devised by Nunn [1913] and by Rugg and Clark [1918].

Problems of the I-2 type, which apply algebraic computations in useful ways, are very rarely used. This may be because teachers think there is no need of applying computation; or it

may be because genuine uses for algebraic computations are not to be found in such matters as a first-year high-school pupil can understand.

The first reason is almost certainly a bad one. Pupils, save some of the very intellectual, are stimulated by seeing what a computative procedure is for, and by associating it with the world outside of mathematics. In arithmetic it is found serviceable to introduce each new item of computational method by some genuine and interesting problem whose solution is facilitated by the computation in question. Students of algebra can doubtless get along without such stimulants better than students of arithmetic, who are younger and duller; and may gain less from them. But they, too, will gain much from such introductory problems showing the service which the computation performs. Again the teacher should examine the problem material of this sort in Nunn and in Rugg and Clark.

The second reason is in part valid. Much of the computational work often done in courses in algebra cannot well be introduced by or related to problems from the world the pupil knows. The problems that one might invent would not make the computation clearer or easier or more esteemed or longer remembered. The value of such computation is questionable.

The miscellaneous group listed as I-3 represents a borderland between what we ordinarily call problems and tasks calling for mathematical inference and conclusions of all sorts. Too little attention has been paid to this group by teachers of algebra. There has been, in fact, so little of such work that we can hardly judge of its value; but it would at least add variety, give more scope for "original" thinking, and assist in integrating a pupil's algebraic abilities into something which for lack of a better term we may call an "algebraic sense"—a somewhat general readiness to see algebraic facts and to think about them with all his algebraic equipment.

The work of II-A represents what is regarded by teachers of the physical and social sciences as the most essential contribution of algebra in preparation for their study. It has the merit that the facts operated with have some chance of being themselves worth thinking about—are not mere valueless items about A's age, or B's time in rowing up a stream, or C's buying and

selling of sheep; and that the relations amongst these facts are likely to be important relations in nature; and that the results obtained are such as a sane man might obtain in that way.

It has also the merits that it emphasizes the fact that a letter can be used to mean any one of the class of numbers that fulfil certain conditions, and that it leads up to the general treatment of the relation of one variable to another.

Two cautions are useful in connection with such work. The first is to be careful not to burden pupils unduly with learning physics or astronomy or engineering for the sake of having genuine formulae. Formulae whose meanings are obvious from a careful reading should be preferred. They should not be complicated by unfamiliar terms, or by the need of difficult inferences to secure consistency in units. Tabular and graphic work may be usefully coordinated with the problem work. The same formulae may often be used in exercises in formula reading and formula framing, evaluation, and the understanding of relations, before they are used in connection with verbal problems.

The problem material from science and engineering that can be adapted so as to satisfy wise teachers of algebra that it is as good for their purposes as the material in customary use about familiar facts like the ages of boys, the hands of clocks, boats on streams, or tanks and pipes, may be rather scanty. The question then arises of using made-up formulae, such, for example, as:

A boy's father gives him each month half as much as the boy earns.

1. Let $e =$ what the boy earns in any month. What will equal what his father gives him for that month?

2. Let $T =$ what the boy gets in all that month. Make a formula for finding T.

3. What will the boy receive in all in each of these months: January, when he earns $10? February, when he earns $12? March, when he earns $8.60?

4. How much must he earn in a month to get (in all) $20 that month?

It is for such cases that we need the second caution, namely, that we avoid in such made-up formulae the unrealities and trivialities that have characterized so much of the problem material of the past.

We have left the II-B type where the equation or formula is not known but must be constructed by the pupil. This is problem material par excellence, and is, in fact, all that certain teachers would consider worthy of the name. Within it, the II-B b type is in far wider use than the II-B a type. In the rest of this chapter, consequently, unless the contrary is specified, we shall mean by a problem one of the type of II-B b where an equation is built up, organizing the data given in the problem about some one state of affairs, so as to secure the answer to one or more quantitative questions about that particular state of affairs.

We shall deal with the following matters, in the order given:

The genuineness of the problem.

The importance of the problem.

Shall every technique be applied to problems?

How far shall the problems be worked as originals, and how far shall routine procedures for solving a certain kind of problem be taught?

The overestimation of the educative value of the verbal problem.

The use of problems at the beginning of a topic to show the need for certain technique and to facilitate the mastery of the technique, as well as at the end to test the ability to apply the technique.

Criteria in selecting problems.

Problems as tests.

Real versus described situations.

Isolated and grouped problems.

Problems requiring the selection of data, as well as their organization.

Problems requiring the discovery of data as well as their selection and organization.

Problems requiring general solutions.

Problems of puzzle and mystery.

The election of problems by students.

Genuineness

Relatively few of the problems now in use are genuine. First of all, over half of them are problems where in the ordinary course of events the data given to secure the answer would them-

selves be secured from the knowledge of the answer. For example, "In ten years John will be half as old as his father. In twenty years he will be three-fifths as old as his father. How old is John now? How old is his father?" In reality such a problem would only occur in the remote contingency that someone knowing that John was 10 and his father 30, figured out these future age ratios, then forgot the original 10 and 30, but remembered what the future ratios were!

We have made the count for three representative textbooks of excellent repute with the results shown in Table 1.

TABLE I

Percentages of "Answer Known" Problems.*

	Up to the beginning of fractions.	The beginning of fractions up to quadratics.	Quadratics and beyond.	Total
Book A	52	69	54	57
Book B	45	36		42
Book C	53	52	51	52

Such problems, if defensible at all, are defensible as mental gymnastics, and as appeals to the interest in mystery and puzzles. As such, they are better if freed from the pretense at reality. "I am thinking of a number. Half of it plus one-third of it exceeds one-fourth of it by seven. What is the number?" is better than problems which falsely pretend to represent sane responses to real issues that life might offer. It is degrading to algebra to put it to work searching for answers which in reality would have been present as the means of framing the problem itself, save frankly as a mere exercise in sharpening one's wits and in translating a paragraph into an equation.

Of the problems which are not clearly ruled out by this criterion many concern situations or questions or both which are not genuine, because in the real world the situation would probable not occur in the way described or because the answer would not be obtained in the way required.

We can make a scale for genuineness running from problems that are fantastic to problems that are entirely genuine. We

*An "Answer Known" problem is one where it is highly probable that in real life the data given would be obtained from the answer rather than the answer from the data. The totals from which the percentages are computed include problems of Type I and Type II-A. If only II-B problems were considered, the percentages of "Answer Known" problems would be much higher.

may set as a very charitable criterion that a problem should be as high as 4 on this scale, 4 being the average genuineness of the following problems.

REALITY 4

A. Three men are asked to contribute to a fund. The first agrees to give twice as much as the second, and the third agrees to give twice as much as the first. How much must each contribute to make a total of $1050?

B. What angle is five times its complement?

C. A boy knows that his boat can go 6 miles per hour with the current and 3 miles per hour against the current. How far can he go and return making the whole trip in just 3 hours?

D. The diagonal of a rectangle is 102 inches and the base of the rectangle is three times its altitude. What is the length of its base?

E. The principal varies directly as the interest and inversely as the rate. If $2000 brings in $125 interest at 4%, how much principal will yield $500 at 5% for the same time?

Reality 4 is obviously not a high standard. It is doubtful whether in all the world's cases of conditional giving the problem of A has ever occured. If B has ever occurred it probably has been in such connections that the immediate solution by $180^\circ - \frac{1}{6}$ of 180° would be used. C illustrates a genuine relation but one which in reality is usually so complicated by other circumstances that only approximate estimate is made; hence the equational treatment seems rather pedantic. It is very hard to conceive cases where a person would know the proportions of a rectangle and the length of its diagonal and not already know the length of the base. The determination of the investment required to yield $500 when you don't know for how long, but do know that $2000 at 4% brings in $125 in the time in question, would not occur probably once in the lifetime of a million men.

We have counted for one of the books noted above the number of problems which passed the "answer known" criterion but failed to rate as high as 4 for genuineness. There were 70 out of 213.

IMPORTANCE

Within the minority of problems that remain after the exclusion of bogus problems and problems whose "genuineness" is less than 4, a considerable percentage concern matters that are of importance to few people and not of much importance to them. The problems of the book in question were rated by four psychologists, three of whom were well versed in mathmatics. Importance 3 means the degree of importance possessed by the following:

IMPORTANCE 3

A. Divide $108 between A and B so that A receives eight times as much as B.

B. A has $2000 invested at 4%. How much must he invest at 6% to make the total yield equal to 5% on the total investment?

C. Mr. A paid $300 per share for some stock. At the end of five years he sold it for $550 per share. What rate of simple interest did his money produce for him during the five years?

D. Mr. A can plow a field in 6 days. Mr. B can plow it in 9 days. How long will it take them to plow the field if they work together?

E. A man does one-third of a piece of work in 10 days. He and another man finish the task together in 8 days. How many days would it take the second man to do the work alone?

F. ABC is an isosceles triangle, AD is its altitude, AD being perpendicular to BC. BD = DC. If AB is 18 inches and BC is 15 inches, find AD.

G. If a boy earns $520 during his first year of work and receives an increase of $50 a year each year thereafter, what salary does he receive the tenth year? How much has he earned in all during the ten years?

H. Each year Mr. A saves half as much again as he saved the year before. If he saved $64 the first year, how much will he save in all in seven years?

If we eliminate all the problems whose importance is less than this, we have left 96, about one-fifth of the total list. Of those so left, many are more suitably solved by mere arithmetical computation without any equation than by the organization around a symbol for the desired number and an equality sign. Omitting these also we have left 61 of the original 491.

These genuine problems which pass our minimum standard of importance for life seem worthy of presentation. The problems themselves show the teacher's available resources in this respect better than a description or tabulation of them would.

Some of these 61 are clearly problems of Type I where no equation or formula is needed (*e. g.*, One bu. equals 32 qt. How many qt. in 4 bu.? In x bu.?) Others are clearly of Type II-A-1 where only substitution in a formula presented at the time is needed (*e. g.*, changing Fahrenheit temperatures to Centigrade). Omitting these, we have left 49, one-tenth of the original series. These 49 problems appear on pages 221 to 223.*

Table II shows the facts separately for the work up to the beginning of fractions, from there up to quadratics, and from quadratics on.

*The problems given here are not quotations, since it seems desirable to preserve the anonymity of the source, but are duplicates of the originals in general nature and form.

TABLE II

Analysis of the Verbal Problems of a Standard Textbook.

	Up to the beginning of fractions.	The beginning of fractions up to quadratics.	Quadratics and beyond.	Total.	
1. Total number of verbal problems	268	126	97	491	
2. Numbers in which the answer would ordinarily not have to be known in order to obtain the data of the problem	129	39	45	213	(43%)
3. Number of these (2) which are rated as 4 or above for reality	100	16	27	143	(29%)
4. Number of these (3) which are rated as 3 or above for importance	73	9	14	96	(20%)
5. Number of these (4) which are not much more readily solvable by arithmetic alone and which are not clearly of Type I (where no explicit equation or formula is needed)	39	9	13	61	(12%)
6. Number of these (5) which are not clearly of Type II-A, where only substitution of numbers in a given formula is required	29	9	11	49	(10%)

An analysis like that of Table II has been made for a second book of excellent repute, which contains work only up to quadratics. It gives the following results:

	Up to fractions.	Fractions to quadratics.	Total.	
1.	283	129	412	
2.	157	82	239	(58%)
3.	125	56	181	(44%)
4.	121	23	144	(35%)
5.	39	19	59	(14%)
6.	38	15	53	(13%)

Eighteen of the problems in this second book left in class 6 are not typical problems of Class II-B. Ten are constructions of graphs, and eight are applications of simple trigonometrical facts not usually taught in algebra hitherto.

The Highest Ranking Tenth of Problems in Respect to Genuineness and Importance

1. The selling price of this book is five-fourths of its cost. Find its cost if it sells for \$2.00.

2. In making a certain casting $1\frac{1}{2}\%$ of the metal is lost in the melting. How much metal is needed to make a casting weighing 86 pounds?

3. Cotton seed meal is used as a fertilizer. It contains approximately 7% of nitrogen. If a farmer wishes to put 15 pounds of nitrogen on a certain field, how much cotton seed meal must be purchased?

4. Tobacco stems contain about 8% of potash. How many pounds of tobacco stems must be bought to obtain 12 pounds of potash?

5. Divide \$108 between A and B so that A receives 8 times as much as B.

6. Three men are asked to contribute to a fund. The first agrees to give twice as much as the second and the third agrees to give twice as much as the first. How much must each contribute to make a total of \$1050?

7. The minimum temperature on February 2nd at Minneapolis was — 15; the maximum was — 4. What was the range of temperature there of that day?

8. Mr. A wishes to enclose a rectangular field, 20 rods wide. He wishes to make the field as long as he can, using 214 rods of fencing. How long can he make it

9. If the cost of a car is p dollars and the rate of gain is 20%, what is the gain? What is the selling price? $(p + .20p = ?)$

10. What was the cost of a car sold for \$13.20 if the gain is 10%?

11. Mr. A wishes to make 20% on some chairs. At what price must he buy them if he is to sell them at \$2.00 each?

12. Mr. B wishes to sell chairs at \$7.00 each. At what price must he buy them so as to make 12% on the cost?

13. Mr. C knows that he can sell a piece of property for \$3540. How much must he pay in order to make a profit of 18%?

14. What principal must be invested at 4% to yield an income of \$600 a year?

15. How long must \$2000 be invested at 5% simple interest to produce \$375 interest?

16. The amount equals the sum of the principal and the interest. Express the amount at the end of a year when p dollars are invested at 4%.

17. What is the amount at the end of two years if p dollars are invested at 5%?

18. What sum of money invested at 6% simple interest for three years will amount to \$4000?

19. How long will it take \$1000 to amount to \$1500 if it is invested at 5%? $(1500 = 1000 + 1000 \times .05y.$ Solve for y.)

20. How long will it take \$1000 to double itself at 6%?

21. Let A represent the number of dollars in the amount. Let P, R and T have their usual meanings. Show that $A = P + \frac{PRT}{100}$

22. Use the formula above to find how many years will be required for \$7000 to amount to \$9100 at 5% simple interest.

23. Use the formula to solve this problem. Mr. A paid \$300 per share for some stock. At the end of 5 years he sold it for \$550 per share. What rate of simple interest did his money produce for him during the five years?

24. Mr. A has tea worth 65 cents and tea worth 45 cents per pound. How many pounds of each should he use to make a mixture of 100 pounds to sell at 53 cents a pound?

25. Mr. B has tea selling at 70 cents a pound and tea selling at 50 cents a pound. How many pounds of each should he use to make a mixture of 50 pounds selling at 62 cents a pound?

26. Same as the two previous, using different numbers.

27. Mr. A has \$2000 invested at 4%. How much must he invest at 6% to make the total yield equal to 5% on the total investment?

28. Mr. B has \$6000 invested at 3½% and \$9000 at 4%. How much must he invest at 6% to make the total yield equal to 5% on the total investment?

29. A boy knows that his boat can go 6 miles per hour with the current and 3 miles per hour against the current. How far can he go and return, making the whole trip in just 3 hours?

30. A man can do a piece of work in 8 days. What part of it can he do in one day? In 7 days? In x days?

31. A man can do a piece of work in x days. What part of it can he do in 1 day? In 5 days?

32. A can do a piece of work in 6 days. B can do it in 10 days. How much can A do in one day? In x days? How much can B do in one day? In x days? How much can A and B together do in one day? In x days? How much can A do in 2 days? How much can B do in 5 days? How much can A and B together do if A works 2 days and B works 5 days?

33. A can do a piece of work in 10 days. B can do it in 5 days. How long will it take A and B together to do it?

34. A can do a piece of work in 8 hours. B can do it in 24 hours. How long will it take A and B together to do it?

35. Mr. A can plow a field in 6 days. Mr. B can plow it in 9 days. How long will it take them to plow it if they work together?

36. One machine can do a piece of work in 4 hours. Another machine can do it in 6 hours. How long will it take them both together to do it?

37. A can do a piece of work in 15 hours. B can do it in 18 hours. If A works for 7 hours, how long will it take B to complete the work?

38. A does one-third of a piece of work in 5 days. A and B complete the job by working together for 4 days. How long would it take B to do the job alone?

39. ABC is an isosceles triangle. AD is its altitude, AD being perpendicular to BC. BD = DC. If AB is 18 inches and BC is 15 inches, find AD.

40. If a boy earns \$520 during his first year of work and receives an increase of \$50 a year each year thereafter, what salary does he receive the tenth year? How much has he earned in all during the ten years?

41. On January 1st of each of 10 years a man invests \$100 at 5% simple interest, what will principal plus interest amount to at the end of the tenth year?

42. Mr. A owes \$2000 and pays 6% interest. At the end of each year he pays \$200 and the interest on the debt which has accrued during the year. How much interest will he have paid off when he has paid off the debt?

43. Mr. A is paying for a \$400 lot at the rate of \$20 a month with interest at 6%. Each month he pays the total interest which has accrued on that month's payment. How much money, including principal and interest, will he have paid when he has freed himself from debt?

44. Mr. A plans to give his son 10 cents on his fifth birthday, 20 cents on his sixth, and each year thereafter to the eighteenth birthday, inclusive, to double the gift of the preceding year. How much will this be in all?

45. A problem in finding the height of a tower by similar triangles. A diagram is given.

46. Finding the width of a pond by similar triangles and subtraction. A diagram is given.

47. Finding the width of a pond by similar triangles and double subtraction. A diagram is given.

48. The number of tiles needed to cover a given surface varies inversely as the length and width of the tile. If it takes 300 tiles 3 inches by 5 inches to cover a certain surface, how many tiles 4 by 6 will be needed for the same area?

49. The number of posts needed for a fence varies inversely as the distance between them. If it takes 120 posts when they are placed 10 feet apart, how many will it take when they are placed 12 feet apart?

It must be confessed that this list of what one of our standard instruments for teaching algebra offers as genuine problems to be solved by framing an equation does not support the general high estimation of problem solving of the II-B-b type. Problems 16 to 23 and 41 will not be acceptable to many because they neglect the fact that in real life the interest on the investment is almost always paid at stated intervals, not when the principal is repaid, and so can be compounded by reinvesment. Nos. 6, 27, 28 and 43 are rather fantastic. Nos. 48 and 49 require a method of finding the number of articles required which would rarely be wise, and never necessary, to use in such situations. Of the other problems, some are very probably better dealt with by the arithmetical methods which the pupils have already learned to use in such cases.

The advocate of the made-up problems will use the scantiness of this list as an argument that we must resort to the made-up, even insane, problems in order to give sufficient practice in applying principles and technique. But why should we give any practice in applying a principle or a technique to created problems when there are no sane problems to which it applies? Moreover, we must not assume that all the problem material which is genuine and of a fair degree of importance has been collected. At first thought, it would seem probable that it had, since for at least a decade progressive teachers and textbook makers have been fully aware of the need for it. A closer study of the matter, however, reveals that ingenuity and inventiveness and careful investigations do bring returns here as elsewhere. Many more such problems appear in the textbooks and teaching of

1920 than were available in 1900. Nunn has made very notable contributions. We may hope that the Nunns of the future will add more. Until we have canvassed the world's work thoroughly for problems that are genuine and important, we ought not to turn to those that are artificial and trivial.

It is a modern tendency to extend the list of genuine problems by teaching certain facts of physics, engineering, astronomy, navigation and the like so as to secure material for practice with the applications of algebra.

We very much need measurements of the time-cost of this, and of its effect upon interest in the sciences in question, in typical cases. The expectation is that often the game is not worth the candle unless it is very skilfully played, and that an undesirable attitude toward science may often result. It should be noted that the experts in teaching science rather carefully avoid algebraic and other quantitative work for pupils in high schools. High-school teachers of chemistry, geology, physical geography, the biological sciences and economics are cautious about employing anything mathematical beyond the simplest; and this partly because they fear that it will repel students. Even in physics, descriptive work is emphasized rather than the fundamental equations; words are used instead of symbols, and sentences instead of formulae. This in spite of the fact that physics is taught in the last or next to last year of high school to a select and mature group. There is a danger that when we select problem material for algebra from the sciences we may be burdening algebra with the least attractive features of science and penalizing science by displaying its least attractive features to the pupil at the beginning of his high-school course.

There has been, so far as I am aware, no direct observational or experimental evidence published concerning the reactions of pupils to these problems taken from the sciences. Nor have we found facilities for securing such. We have, however, secured the judgments of the four psychologists mentioned previously.

They rated seventeen such problems (11 about the principal that weight times length of lever arm equals weight times length of lever arm to make a balance, 5 about freely falling bodies, and 1 about the pressure-volume relation in gases) for reality, importance, interest, value in showing and in applying mathematical laws, excellence of statement, and value in teaching facts

or laws outside of mathematics. In the combined weighted average, these problems from physics were somewhat above the average of problems in present-day textbooks.

Selection of Techniques for Application

It is hard to find psychological or pedagogical justification for the custom of concluding each topic in algebra by a series of verbal problems whose solution requires the operation of the mathematics taught under that head. The custom seems to be due partly to habits carried over from arithmetic, partly to the general fondness of intellectual persons for neat symmetrical systems, partly to a general overvaluation of the verbal problem as a means of mental training, and partly apparently to an insufficient appreciation of pure mathematics itself.

It is not likely that the arrangement of problems applying mathematical technique which is best for all children in grades 3 to 6 will be the best for the third of them who go on to study algebra in grade 9. Nor is it at all certain that "technique—application—technique—application" is the best arrangement in grades 3 to 6. Good practice in the teaching of arithmetic now supplements this arrangement in grades 3 to 6 by an arrangement by topics like "Earning and Saving," "Distances in a City," "House Plans," "A School Garden." In grades 7 and 8, the arrangement has long been largely by topics like Insurance, Investments, Interest Given by Savings Banks, and Bank Loans, and the like, and is developing toward an arrangement by topics like Food Values, City Expenditures, and Wage Scales.

Whatever arguments may be derived from the advantages of system would seem to be in favor of giving the main treatment of problem solving in one large unit, the general task of which would be to show that any number or numbers which can be found from certain given data, can be found by expressing the proper data in an equation or equations and solving. This chapter could well come after the pupil had learned to add, subtract, multiply and divide with literal numbers, including such simple fractions as should be mastered, and before the systematic treatment of the relation $y = ax + b$, or any treatment of quadratic equations. If a problem is suggested that leads to a quadratic (or a cubic), no harm will be done. The pupil may frame the equation, and leave it for solution until he learns the tech-

nique. This matter will be discussed further as one special problem of the order of topics in algebra. Our present purpose is simply to suggest that system does not require, or even favor, applying every technique indiscriminately in verbal problems. Of the general over-valuation of verbal problems and under-valuation of pure mathematics we shall treat in detail later.

All these are matters of minor importance for our present question if we accept as true a proposition which seems to the psychologist almost indubitable; namely, that the peculiar educative values of these verbal problems are attained by *framing* the right equations, solving them being not very greatly different from solving a similar equation framed for you by the textbook. If the problems are given primarily to train the pupil to frame the right equation or equations, we care very little about what computational techniques they happen to lead to. To take the extreme case, suppose that pupils only framed and *never* solved the equations, as in the Hotz test for problem solving. It would then be of almost no importance which techniques were required in these solutions—whether, for example, abilities with surds, quadratic equations, and certain factorizations were or were not applied. In so far as the peculiar value of problems is in framing the equations it is better *not* to give, after each technique is learned, many of the problems applying to it, because this tempts the pupil to expect that the problems will have a certain sort of equational form. He is tempted to work *toward* a certain sort of equation, instead of *from* the data given. It would then be amongst "miscellaneous" that a problem usually gave its best training.

We do not mean to imply that the framing of an equation and its solution are as educative if done a month apart as if done together, or that solving equations already framed for you is as educative as solving equations which you have framed yourself. We do claim that the peculiar virtue of the verbal problem is in the framing, not the solving, and that problems should be selected and arranged from this point of view rather than as exercises to show that certain algebraic computational tasks can be used in problems and to give practice in their use.

Problems as Originals and as Semi-Routines

The guiding principles in relation to this question can be briefly stated as follows:

Other things being equal, it is more educative to solve a problem as an original. Individual differences in ability need most of all to be allowed for when problems are given as originals. It is not probable that a pupil's efforts to solve problems are of great value to him when he fails with more than two out of three of them. On the other hand, pupils who are able to solve a certain type of problem as an original should certainly be excused from training in a routine method of solving it. Special training in the method of solving a certain type of problem is not desirable unless the problems in question are genuine and of some considerable importance. Clock, digit, age and other similar problems should be given as originals, if at all. Whatever view we take of the amount of general ability developed by problem solving, one of the best ways to develop it is by trying to solve problems as originals, and, in case of failure after a reasonable effort, being given such assistance as enables one to solve them.

(To be concluded in the May issue.)